Perturbed solution of sea-air oscillator for the El Niño/La Niña-Southern Oscillation mechanism

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Received 13 November 2006; accepted 5 January 2009

Abstract

A class of coupled system of the El Niño/La Niña-Southern Oscillation (ENSO) mechanism is studied. Using the perturbed theory, the asymptotic expansions of the solution for ENSO model are obtained and the asymptotic behavior of solution for corresponding problem is considered.

Key words: nonlinear, perturbation theory, El Niño/La Niña-Southern Oscillation model

1 Introduction

El Niño/La Niña and Southern Oscillation (ENSO) are the abnormal events happening in the atmosphere and ocean of tropical Pacific. ENSO is a very attractive phenomenon in the international academic circles (Huang et al., 2004; Feng et al., 2002; Liu et al., 2002; Wang, 2001). Lin and Mo (2003) considered a class of problems in atmospheric physics, ocean weather, dynamic system and so on (Lin, Ji, Wang, et al., 2000; Lin, Ji, Li et al., 2000; Lin et al., 2001; Lin, Ji, Wang, 2002; Lin, Ji, Wang et al., 2002; Lin and Mo, 2003; Mo and Lin, 2005, 2004; Mo, Lin and Zhu, 2004a, b, c). The oscillatory nature of ENSO requires both positive and negative ocean-atmosphere feedbacks. And a positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific is considered. This anomaly reduces the zonal SST gradient and the strength of the Southern Oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation changes and then reinforce SST anomaly.

Recently many scholars have investigated the nonlinear perturbed problem. Approximation methods have been developed and refined, including the method of averaging, boundary layer method, the matched asymptotic expansion method, and the multiple scales method (de Jager and Jiang, 1996; Hamouda, 2002; Bell and Deng, 2003; Adams et al., 2003). Using the perturbed method, Mo and Lin et al. considered a class of perturbed problems (Mo, Lin and Zhu et al., 2004d; Mo et al., 2003; Mo and Wang, 2002; Mo et al., 2002; Mo, 2001, 1989). In this paper we study a class of ENSO model by using a simple and special method for perturbed theory.

The ENSO oscillator model should consider the variations of both the eastern and western Pacific anomaly patterns. From complexity of ENSO, then we only study that the some oscillator model overlooks the ENSO western Pacific anomaly patterns and does not consider the effect of the western Pacific in ENSO. As it will be shown below, the oscillator model can be reduced to a simple oscillator model by further simplification and assumptions.
2 ENSO model

It is assumed that winds in the western Pacific do not affect the SST anomalies in the eastern Pacific for 10°S–10°N, 140°W–85°W. Thus the coupled system requires equations as follows (Huang et al., 2004)

\[
\begin{align*}
\frac{dT_a}{dt} &= a_{11}T_a + a_{12}T_s + \varepsilon b_{11}T_aT_s + \varepsilon b_{12}T_s^2, \\
\frac{dT_s}{dt} &= a_{21}T_a + a_{22}T_s + \varepsilon b_{21}T_aT_s + \varepsilon b_{22}T_s^2,
\end{align*}
\]

(1)

(2)

where \( T_a \) is the locally averaged temperature anomaly of air; \( T_s \) is the locally averaged temperature anomaly of the sea surface in the equatorial eastern Pacific; \( \varepsilon \) is a small positive perturbed parameter; coefficients \( a_{ij} \) and \( \varepsilon b_{ij} \) are real non-negative constants (0.04, 0.12) determined by the absorption rate of in-cloud liquid water, the densities (1.25 kg/m³, 100 kg/m³) of the troposphere and sea mixed layer, Bowen ratio (3), the thermocline depth (100 m), mean vertical velocity \( (5 \times 10^{-6} \text{ m/s}) \) in the thermoclin, Coriolis parameter \( (10^{-5} \text{s}^{-1}) \), the coefficients of sensible heat exchange \( (0.001 \text{ m}^2/\text{s}) \) and eddy viscosity \( (5 \text{ m}^2/\text{s}) \) and so on, which expansions see Huang et al. (2004) and Xiang (1990).

3 Approximate solution and its precision

Let the solution \( [T_a(t, \varepsilon), T_s(t, \varepsilon)] \) of the nonlinear coupled systems (1) and (2) be

\[
\begin{align*}
T_a(t, \varepsilon) &= \sum_{i=0}^{\infty} T_{ai}(t) \varepsilon^i, \\
T_s(t, \varepsilon) &= \sum_{i=0}^{\infty} T_{si}(t) \varepsilon^i.
\end{align*}
\]

(3)

(4)

Substituting Eqs (3) and (4) into Eqs (1) and (2), let \( \varepsilon = 0 \), we have the reduced system of original coupled systems (1) and (2)

\[
\begin{align*}
\frac{dT_{a0}}{dt} &= a_{11}T_{a0} + a_{12}T_{s0}, \\
\frac{dT_{s0}}{dt} &= a_{21}T_{a0} + a_{22}T_{s0}.
\end{align*}
\]

(5)

(6)

It is not difficult to see that the characteristic roots \( \lambda_i, i=1,2 \) of a linear systems (5) and (6) are

\[
\begin{align*}
\lambda_1 &= \frac{1}{2}[(a_{11} + a_{22}) - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}], \\
\lambda_2 &= \frac{1}{2}[(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}].
\end{align*}
\]

(7)

(8)

Obviously, as \( a_{11}a_{22} < a_{12}a_{21} \), we can decide that the two characteristic roots from Eqs (7) and (8) are real different signs: \( \lambda_1 < 0 < \lambda_2 \), thus from the theory of singular point, the zero point in the phase plane of the linear system (5) and (6) is an instable saddle point; as \( a_{11}a_{22} > a_{12}a_{21} \), the two characteristic roots are different positive real part roots, thus there is an instable node or focus and as \( a_{11}a_{22} = a_{12}a_{21} \), there are a positive characteristic root and a zero root, thus there exists a singular line in the phase plane and the singular line is instable too. And the trajectories of the phase plane for the system are away from the zero point.

Because the nonlinear terms of the coupled systems (1) and (2) are \( R_1 = \varepsilon(b_{11}T_aT_s + b_{12}T_s^2) \) and \( R_2 = \varepsilon(b_{21}T_aT_s + b_{22}T_s^2) \), then for \( 0 < \varepsilon \ll 0 \), we have

\[
\lim_{\|y\| \to 0} \frac{\|R\|}{\|y\|} = 0,
\]

where

\[
\|R\| = (R_1^2 + R_2^2)^{1/2}, \quad \|y\| = (T_a^2 + T_s^2)^{1/2}.
\]

Thus there is same stable behavior for the solution of the nonlinear coupled systems (1) and (2) and the linear systems (5) and (6). Therefore, the solution of the systems (1) and (2) is unstable.

Substituting Eqs (3) and (4) into Eqs (1) and (2), comparing the coefficients of \( \varepsilon^i \), we also have

\[
\begin{align*}
\frac{dT_{a1}}{dt} &= a_{11}T_{a1} + a_{12}T_{s1}, \\
&\quad + b_{11}T_{a0}T_{s0} + b_{12}T_{s0}^2, \\
\frac{dT_{s1}}{dt} &= a_{21}T_{a1} + a_{22}T_{s1} + b_{21}T_{a0}T_{s0} + b_{22}T_{s0}^2.
\end{align*}
\]

(9)

(10)

From Eqs (9) and (10), we can obtain a solution \( [T_{a1}(t), T_{s1}(t)] \).

Substituting Eqs (3) and (4) into Eqs (1) and (2), for the coefficients of \( \varepsilon^i, i = 2, 3, \cdots \) we also have

\[
\begin{align*}
\frac{dT_{a2}}{dt} &= a_{11}T_{a2} + a_{12}T_{a1} + F_{1i}, i = 2, 3, \cdots, \\
\frac{dT_{s2}}{dt} &= a_{21}T_{a2} + a_{22}T_{s1} + F_{2i}, i = 2, 3, \cdots,
\end{align*}
\]

(11)

(12)

where \( F_{ji}, j = 1, 2, i = 2, 3, \cdots \) are determined functions for \( i \) successively. And their constructions are omitted. From linear non-homogeneous coupled systems (11) and (12), we also can obtain the solutions \( T_{a1}(t), T_{s1}(t), i = 2, 3, \cdots \) successively. We then can
find the asymptotic expansions (3) and (4) of the solution for the coupled systems (1) and (2).

And let

$$T_a(t, \varepsilon) = \sum_{i=0}^{n} T_{ai}(t) \varepsilon^i + R_1(t, \varepsilon),$$

$$T_s(t, \varepsilon) = \sum_{i=0}^{n} T_{si}(t) \varepsilon^i + R_2(t, \varepsilon).$$

Substituting Eqs (13) and (14) into Eqs (1) and (2), we yield

$$\frac{dR_1}{dt} = a_{11}R_1 + a_{12}R_2 + O(\varepsilon^{n+1}), 0 < \varepsilon \ll 1,$$

$$\frac{dR_2}{dt} = a_{21}R_1 + a_{22}R_2 + O(\varepsilon^{n+1}), 0 < \varepsilon \ll 1.$$  

From Eqs (15) and (16), it is easy to see that

$$R_i(t, \varepsilon) = O(\varepsilon^{n+1}), i = 1, 2, 0 < \varepsilon \ll 1.$$ 

Thus we have that Eqs (3) and (4) are uniformly valid asymptotic expansions of the solution for the nonlinear coupled systems (1) and (2).

4 Discussion

(1) From the above conclusion, we know that the sea-air oscillator of an ENSO model (1) and (2), the air temperature and sea surface temperature anomalies in the equatorial eastern Pacific are unstable trend and possess asymptotic unstable annual phenomenon.

(2) We consider roughly the precision of the above results (13) and (14). As $0 < \varepsilon \ll 1$, it is easy to see that we can obtain the asymptotic solution $T_a(t, \varepsilon), T_s(t, \varepsilon)$ for the original model (1) and (2) using the perturbed method as follows:

$$T_a(t, \varepsilon) = \sum_{i=0}^{n} T_{ai}(t) \varepsilon^i + O(\varepsilon^{n+1}), 0 < \varepsilon \ll 1,$$

$$T_s(t, \varepsilon) = \sum_{i=0}^{n} T_{si}(t) \varepsilon^i + O(\varepsilon^{n+1}), 0 < \varepsilon \ll 1.$$ 

(3) Atmospheric physics is a very complicated natural phenomena. Hence we need to reduce basic models for the sea-air oscillator. And then we solve them using the approximate method. The perturbed method is a simple and valid method.

(4) The perturbed method is an approximate analytic method, which differ from general numerical method. The expansions of solution using the perturbed method can be continuously analytic operation. Thus, from Eqs (17) and (18), we can study further that the qualitative and quantitative behaviors of the temperature anomaly in the equatorial eastern Pacific, the thermocline anomaly and the zonal wind stress anomaly. But in this paper we do not discuss.

References


