The difference between the joint probability distributions of apparent wave heights and periods and individual wave heights and periods

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Received 3 March 2004; accepted 20 May 2004

Abstract

The joint distribution of wave heights and periods of individual waves is usually approximated by the joint distribution of apparent wave heights and periods. However there is difference between them. This difference is addressed and the theoretical joint distributions of apparent wave heights and periods due to Longuet–Higgins and Sun are modified to give more reasonable representations of the joint distribution of wave heights and periods of individual waves. The modification has overcome an inherent drawback of these joint PDFs that the mean wave period is infinite. A comparison is made between the modified formulae and the field data of Goda, which shows that the new formulae consist with the measurement better than their original counterparts.

Key words: sea waves, wave statistics, wave height, wave period

1 Introduction

The statistics of sea waves is essential for ocean engineering, as well as for marine science. In the past few decades, great effort has been put on this topic. Longuet–Higgins (1952, 1975, 1983), Sun (1987, 1988), Xu et al. (2000), Guan (1998), and others, studied the statistics of narrow-banded linear waves. Lindgren (1972), Cavanaugh et al. (1976), Lindgren and Rychlik (1982), Shao and Yu (1987), and others, investigated the statistics about the surface maxima of broad-banded linear sea waves. Further, Kingsman (1965), Longuet–Higgins (1963), Tayfun (1980), Huang et al. (1983), Hou (1990), Sun and Ding (1994), Guan and Sun (1997), and others, extended the statistics to non-Gaussian waves. Besides the above theoretical researches, there are also a lot of experimental studies and data analyzing work, such as Bretschneider (1959), Chakrabarti and Cooley (1977), Goda (1978), Forristall (1978), Huang and Long (1980), Ding et al. (1995a, b), Wu et al. (1981), Huang et al. (1995), etc. These works have also yielded several empirical probability distributions.

In this paper we discuss about the joint probability distribution function (PDF) of wave heights and periods $P(T, H)$. Based on a narrow-banded approximation applied to the linear

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ocean wave model, Longuet–Higgins (1975) represented wave amplitude and period by the wave envelope function and the time derivative of the phase function respectively [see Eq. (1) in the next section] and derived a theoretical $P(T, H)$. This joint PDF fitted well with the data of Bretschneider (1959) for narrow-banded spectrum; however, it failed to account for the asymmetry in the distribution of wave period that is commonly observed when the spectral width is relatively broad. Longuet–Higgins (1983) improved his theory through introducing a normalization factor. Although the improved joint PDF became asymmetrical, it yielded a slightly non-Rayleigh wave height distribution. Sun (1987) derived an alternative form of $P(T, H)$ under the assumption that the ray theory is applicable to linear sea waves. There is no discernible difference between the joint PDF of Longuet–Higgins (1983) and that of Sun (1987), except that the latter yielded a Rayleigh-form PDF for wave heights. Lindgren (1972), Cavanié et al. (1976) and Lindgren and Rychlik (1982) each gave a theoretical form of $P(T, H)$. We do not discuss these distributions since the wave height and period in them were defined differently from Longuet–Higgins (1975), and moreover, they involve the high order spectral moments, which makes them inconvenient for application. The empirical forms of $P(T, H)$ proposed by Wu et al. (1981) and Huang et al. (1995) are not discussed in this paper either, since they are either for shallow water waves or for waves with double-peaked spectrum.

The joint PDF $P(T, H)$ of Longuet–Higgins (1983) as well as that of Sun (1987) are quite successful for narrow-banded wind waves in deep water (Chakrabarti and Cooley 1977; Goda 1978). However there are still some problems lying in them. Firstly, at relatively broad spectral widths, they deviate significantly from the data of Goda (1978). From Figs 10 and 11 in Longuet–Higgins (1983), one may see that, at relatively broad spectral widths, the maximum probabilities occur theoretically around $(T = T_m, H = H_m)$, while Goda’s (1978) measurement shows that the most probable $T$ and $H$ are much nearer to the origin. Here $T_m$ and $H_m$ denote the average wave period and height respectively. Secondly, the mean period yielded from either the joint PDF $P(T, H)$ of Longuet–Higgins (1983) or that of Sun (1987) is infinite, i.e., $\bar{T} \to \infty$, therefore Sun (1987) had to estimate it by the average of $T^{-1}$.

The joint PDF $P(T, H)$ of Longuet–Higgins (1983) and that of Sun (1987) are joint PDFs of the apparent wave heights and periods; hence they are approximations of the joint PDF of wave heights and periods of individual waves. The wave height (or period) of an individual wave is usually defined as the vertical distance between the maximum and the minimum that are within two successive zero-upcrossings (or the time interval between two successive zero-upcrossings). The discrepancy of the joint PDF $P(T, H)$ of Longuet–Higgins (1983) or Sun (1987) with field data may come from several aspects, such as the nonlinear effect, the sampling rate, the difference between the continuity of wave envelopes and phase functions and the discreteness of wave heights and periods of individual waves, etc. The effects of some of these aspects on wave height statistics have been investigated, for instance, Ohta and Kimura (1994), as well as Kitano and Mase (1998), studied the gaps between wave heights and the envelopes, Stansell et al. (2002) the effect of the sampling rate. As far as we know, there is no work on the difference between the joint PDF of apparent wave heights and periods and that of individual wave heights and periods. This aspect of difference is examined in this paper, and accordingly, a modification to the joint PDF of Longuet–Higgins (1983), as well as that of Sun

(1987) is proposed to give more reasonable representations of the joint PDF of the wave heights and periods of individual waves. After that, the modified joint PDFs will be compared with Goda’s (1978) data.

The work of this paper not only is meaningful to the prediction of the joint distribution of wave heights and periods of individual waves, but also may benefit the prediction of the probability distributions of wave heights, wave periods and wave steepness, since the latter three PDFs may be derived from the former. In addition, in Sun (1988), which is an extension of Sun (1987) theory to the three-dimensional wind waves, the joint PDF of apparent wave heights, periods and wavelengths shall also be modified in a way similar to this paper to make it represent more appropriately the joint PDF of the corresponding wave parameters of individual waves. Furthermore, the works making use of the joint PDFs of Longuet–Higgins (1983) or Sun (1987, 1988), such as Liu et al. (1997), Brodtkorb et al. (2000), Xu et al.(2000), Yu and Xu. (1997), Zheng et al.(1999), etc., may also be improved with reference to this paper.

2 The joint PDF of apparent wave heights and periods

In this section we reproduce briefly the theory of Longuet–Higgins (1983) and Sun (1987) to facilitate the discussion of the next section. Assuming the sea waves are narrow-banded and linear, Longuet–Higgins (1983) represented the two-dimensional surface wave elevation \( \zeta (t) \) in the form

\[ \zeta (t) = \text{Re} \rho \exp(i\phi) \exp(i\bar{\omega}t) , \]

where \( \rho (t) \) and \( \phi (t) \) are the apparent wave amplitude and phase function respectively; \( \bar{\omega} \) is the carrier frequency chosen to be

\[ \bar{\omega} = m_i/m_0 , \]

in which \( m_i \) is the \( i \)th order moment of the spectral density \( S (\omega) \) of the process \( \zeta (t) \):

\[ m_i = \int_0^\infty \omega^i S (\omega) d\omega . \]  (3)

He defined the apparent wave height \( H \) and the apparent wave period \( T \) as

\[ H = 2\rho, \quad T = 2\pi/(\omega + \phi) \]  (4)

where the dot upon the phase function denotes differenticn with respect to \( t \). He further found the joint PDF of their nomalized counterparts to be

\[ P_{\rho_0} (h, \tau) = \left( \frac{2}{\pi^{3/2}} \right) \left( \frac{h^2}{\tau^2} \right) L (\nu) \times \exp \left\{ -h^2 \left[ 1 + (1 - 1/\tau)^2 / \nu^2 \right] \right\} , \]

(5)

where

\[ h = H/(8m_0)^{1/2}, \quad \tau = T/(2\pi m_0/m_1) ; \]

the spectral width \( \nu \) is defined as

\[ \nu^2 = \frac{m_0 m_2}{m_1^2} - 1 ; \]

and \( L (\nu) \) is a normalization factor introduced to take account of the fact that only the positive values of \( \tau \) are considered:

\[ \frac{1}{L} = \int_0^\infty \int_0^\infty P_{\rho_0} (h, \tau) \, dh d\tau . \]

(8)

On the other hand, Sun (1987) assumed that the ray theory is applicable to ocean waves with slowly varying wave amplitudes and periods, and introduced the apparent wave amplitude \( \rho (t) \) and the phase function \( \phi (t) \) in an alternative way:

\[ \zeta (t) = \rho (t) e^{i\phi (t)} , \]

then defined the apparent wave height and period as

\[ H = 2\rho, \quad T = 2\pi/|\phi| \]  (9)

Normalizing \( H \) and \( T \) in the same way as Eq. (6), the joint PDF due to Sun (1988) may be rewritten as

\[ P_{\rho_0} (h, \tau) = \left( \frac{2}{\pi^{3/2}} \right) \left( \frac{h^2}{\tau^2} \right) \times \left[ 1 + \exp \left( \frac{4h^2}{\nu^2 \tau} \right) \right] \times \exp \left\{ -h^2 \left[ 1 + (1 - 1/\tau)^2 / \nu^2 \right] \right\} . \]

(10)

Note that the joint PDF \( P_{\rho_0} (h, \tau) \) yields a
Rayleigh distribution for wave height, while the joint PDF $P_{a}(h, \tau)$ yields a slightly non-Rayleigh one. Sun (1988) compared $P_{as}(h, \tau)$ with $P_{a}(h, \tau)$ and found the two were almost the same. Therefore, in the following section, we will only compare $P_{as}(h, \tau)$ with its modified counterpart.

3 The difference between the joint PDFs of apparent and individual wave heights and periods

Equation (5) (or (11)) is often taken as an approximation of the joint PDF of the wave heights and periods of individual waves. However they are not identical. In fact, for the surface elevation $\zeta(t)$, the probability $P_{a}(H, T) \times dHdT$, in which the joint PDF $P_{a}(H, T)$ is given by Eq. (5) or (11), shall be interpreted as the probability that at a given time $t$ the apparent wave height and period simultaneously fall in the range $(H-H+dH, T-T+dT)$, rather than the probability that the wave height and period of a particular individual wave fall in the same range. We denote the latter by $P_{i}(H, T) \times dHdT$.

To find the relationship between the two Joint PDFs $P_{a}(H, T)$ and $P_{i}(H, T)$, let us consider the surface elevation process $\zeta(t)$, of which the duration $D$, as well as the overall number of individual waves $J$ are assumed to be sufficient for the following statistics. For narrow-banded waves with slowly varying wave heights and periods, it is common to assume that the values of the apparent wave height and period are constant over the time scale of an individual wave period and to take them as the wave height and period of that individual wave. Let $\delta j$ denote the number of individual waves with wave heights and periods simultaneously falling in the range $(H-H+dH, T-T+dT)$, then the probability $P_{i}(H, T) \times dHdT$ shall be

$$P_{i}(H, T) \times dHdT = \delta j(H, T)/J.$$  \hspace{1cm} (12)

Assuming that there are overall $J'$ sample points $\{t_{i}\} (i=1, 2, 3, \ldots, J')$ distributed uniformly within the duration $D$ of the process $\zeta(t)$, among which there are $\delta j'(H, T)$ points where the apparent wave height and period simultaneously fall in the range $(H-H+dH, T-T+dT)$, then

$$P_{a}(H, T) \times dHdT = \delta j'(H, T)/J'. \hspace{1cm} (13)$$

Since

$$\frac{\delta j(H, T)}{J} = \frac{\delta j'(H, T) T_{av}}{J'T}, \hspace{1cm} (14)$$

from Eqs (12) to (14), one easily gets

$$P_{i}(H, T) = \frac{T_{av}}{T} P_{a}(H, T), \hspace{1cm} (15)$$

where the mean period of individual waves may be represented by the well-known mean zero-crossing period

$$T_{av} = 2\pi(m_{a}/m_{z})^{1/2}. \hspace{1cm} (16)$$

Therefore, Eqs (5) and (11) shall be modified as follows to give more reasonable representation of the joint PDFs of the wave heights and periods of individual waves:

$$P_{il}(h, \tau) = (v^{2} + 1)^{-1/2} \left(2/\pi v^{2}\right) \left(h^{2}/\tau^{2}\right) \times \exp \left[-h^{2} \left[1 + (1/\tau)^{2}/v^{2}\right] \right], \hspace{1cm} (17)$$

and

$$P_{is}(h, \tau) = (v^{2} + 1)^{-1/2} \left(2/\pi v^{2}\right) \left(h^{2}/\tau^{2}\right) \times \left[1 + \exp \left(4h^{2}/v^{2}\tau\right) \times \exp \left(-h^{2} \left[1 + (1/\tau)^{2}/v^{2}\right] \right) \right]. \hspace{1cm} (18)$$

At the infinitely small spectral width $v \rightarrow 0$, the difference between $P_{i}(H, T)$ and $P_{a}(H, T)$ would disappear. However, the real sea is never like that. For ocean waves with relatively broad spectral widths, this difference shall not be overlooked. For several values of spectral width, numerical calculation according to Eqs (11) and (18) are illustrated in Fig. 1.

Another advantage of the modification is that the mean period can be calculated from Eqs (17) or (18) in a usual way:
\[
\bar{T} = \int_0^\infty \int_0^\infty TP(H,T)dHdT = \\
= \int_0^\infty \int_0^\infty T \frac{T}{T_p} P(H,T)dHdT = T_{av}.
\]

It is worth noting that the above analysis may be extended to the joint PDF of wave heights and wavelengths. For the surface elevation \( \zeta(x) \), let \( P_1(H, \lambda) \) and \( P_a(H, \lambda) \) denote respectively the joint PDF of individual and apparent wave heights and wavelengths, similar analysis yields

\[
P_1(H, \lambda) = \frac{\lambda_{av}}{\lambda} P_a(H, \lambda),
\]

where \( \lambda_{av} \) is the mean individual wavelength, which may be represented by the mean zero-crossing space period.

4 Comparison with published data

Goda (1978) presented diagrams of the relative wave height \( H/H_{av} \) against the relative wave period \( T/T_{av} \), as in Fig. 1, where \( H_{av} \) is the mean individual wave height. The average values of the spectral widths corresponding to these diagrams were estimated by Longuet–Higgins (1983). Longuet–Higgins (1983) compared Eq. (5) with these data. He remarked: "The agreement is reasonable, except that the theory is consistently lower. Most of the discrepancy seems due to our choice of \( T_{av} \)--the position of the modes agrees fairly well, though--there is an indication that for the large values of \( v \) the mode of the distribution (measurement) splits into two, one farther from and the other nearer to the origin."

To compare Eq. (17) with Goda's (1978) measurement, one has to estimate the values of \( H_{av} \) and \( T_{av} \) used by Goda (1978). Following Longuet–Higgins (1983), we assume the average wave period \( T_{av} \) to be given by Eq. (16). Since the theoretical and the experimental PDFs of wave height distributions are either Rayleigh or slightly non-Rayleigh, we estimate the mean wave height \( H_{av} \) by the formula \( H_{av} = \sqrt{2\pi n_0} \).

Figure 1 shows that Eq. (18) agrees with Goda's (1978) measurement better than Eq. (11), in particular, at broad band-widths (Figs 1c and d), the position of the contours of Eq. (18) corresponding to probabilities close to unit fit better with the data than those of Eq. (11). Note that the most probable contour of Eq. (11) surrounds a center roughly at (1,1), no matter how much the spectral width is. The modification makes these contours move towards the origin as the band-width increases, which is in consistent with the data.

The contours of Eqs (11) and (18) consistently deviate leftward from the data. We agree with Longuet–Higgins' (1983) remark that this discrepancy seems mainly due to our choice of \( H_{av} \) and \( T_{av} \). Multiplying \( H_{av} \) or dividing \( T_{av} \) by a coefficient of value approximately 1.1 would make the position of theoretical contours fit well with the data. It is quite possible that such a coefficient exists between what we choose and what Goda (1978) gets from the field data.

At very low values of \( T/T_{av} \), the agreement of Eq. (18) data is not good. Notice that the sampling rate of the data is about 0.25\( T_{av} \). Such a low sampling rate makes the data corresponding to \( T/T_{av} \) being less than 0.5 doubtful.

Additionally, the shape of the contours of Eq. (18) of probability close to 1 may be an indication of the splitting of the modes noticed by Longuet–Higgins (1983).

5 Conclusions

This paper addresses the difference between the joint PDFs of the apparent wave heights and periods and the individual ones. Based on the analysis of the difference, the theoretical joint PDF of the apparent wave heights and periods due to Longuet–Higgins (1983), as well as that due to Sun (1988) are modified to give more reasonable representations of the joint PDF of
Fig. 1. Contours of (solid lines) and (dashed lines) for values of corresponding to Goda’s (1978) data (the numbers in the diagrams). The contours take the values 1.0, 0.5, 0.1 respectively from the center curve outwards.

The contours take values 1.0, 0.5, 0.1 in a and b, 0.8, 0.5, 0.1 in c and 0.7, 0.5, 0.1 in d from inner to outer.
the individual wave heights and periods. There is significant difference between them at relatively broad band widths, which exhibits itself particularly in the position of the contours corresponding to large probabilities. The modified joint PDFs mainly have two advantages: (1) the mean wave period can be calculated from them in a usual way; (2) the position of the contours fits better with the data, in particular, at large band widths the most probable contours move towards the origin, in agreement with Goda's (1978) data.

Acknowledgements

This work was supported by the National “863” High Technology. Project of China under contract No. 2003AA639380.

References


